ETMAG

CORONALECTURE 3

Functions, elementary functions

We begin 12:15

FUNCTIONS

We consider functions $f: X \to Y$ where X and Y are subsets of $\mathbb{R}$. Usually we choose X as large as possible (the so-called *natural domain* of the function *f*).

It is often convenient to choose Y = *f*(X). Then *f* is a surjection (*onto*) and Y is called the *set of values* for *f*.

Everything said in EIDMA lecture about functions in general applies here.

Monotonic functions are defined in the same way as monotonic sequences. In fact, every sequence is a function, so it would be better to define monotonic functions first. But we covered sequences first, so …

Definition.

A function is called *constant on A*, $A \subseteq X$, iff

$$(\exists\ c \in \mathbb{R})\ (\forall\ x \in A)\ f(x) = c$$

Definition.

A function $f : X \to Y$ is called *increasing on A*, $A \subseteq X$, iff

$$(\forall\ x,y \in A)\ (\ x < y \Rightarrow f(x) < (f(y)\ )$$

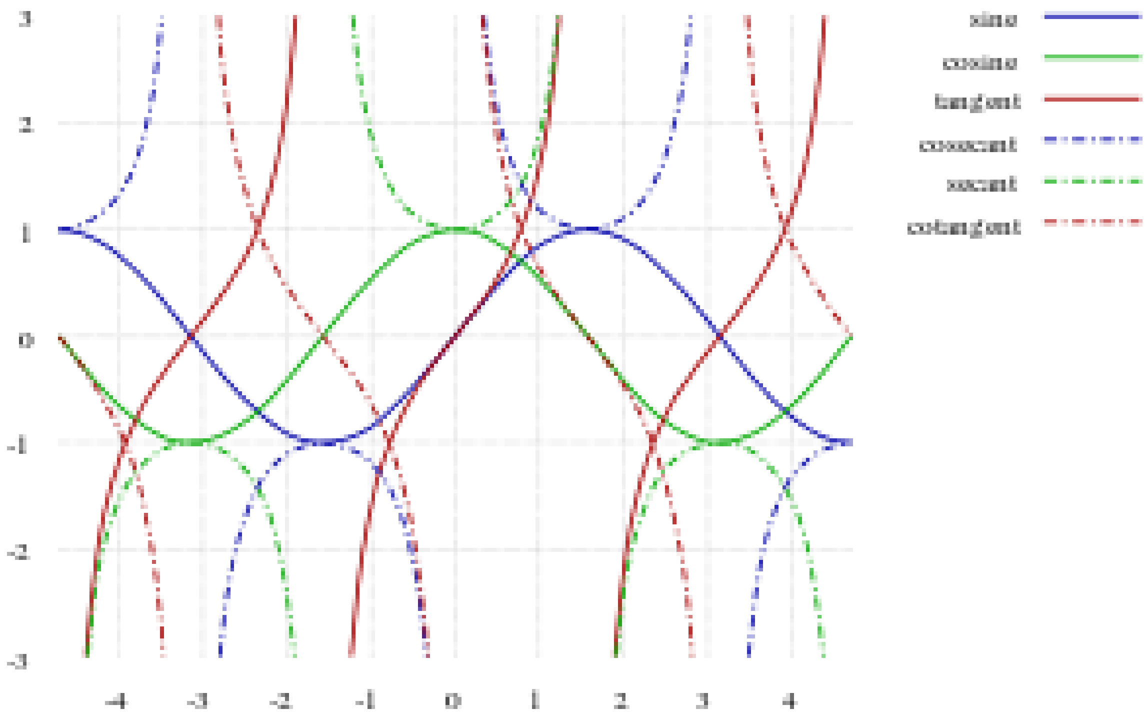
In a similar way we define *nondecreasing* functions.

Example.

The function floor, $\lfloor x \rfloor$ = the largest integer l s.t. $l \leq x$ is globally nondecreasing. It is constant on every interval of the form $[k;k+1]$.

Example.

Consider *tan* x. The domain (or natural domain), of *tan* x is $\mathbb{R} \setminus \{k\pi + \frac{\pi}{2} \mid k \in \mathbb{Z}\}$. *tan x* is increasing on every interval (a,b) which is contained in its domain but it is not increasing *globally.* (graph from Wikipedia).



Definition.

A function is called *decreasing on A*, $A \subseteq X$, iff

$$(\forall\, x,y \in A)\ (\ x < y \Rightarrow f(x) > (f(y)\ )$$

In a similar way we define *nonincreasing* functions.

Example.

Sine is increasing on every closed interval of the form $[2k\pi - \frac{\pi}{2}; 2k\pi + \frac{\pi}{2}]$ and decreasing on every closed interval of the form $[2k\pi + \frac{\pi}{2}; 2k\pi - \frac{\pi}{2}]$, where k is an integer.

Comprehension.

What can you say about a set A and a function f if

- f is at the same time nonincreasing and nondecreasing on A
- f is at the same time increasing and decreasing on A

Definition. (*Reminder*)

Given a function $f : X \to Y$, if there exists a function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$ then it is called the *inverse function* for f or f*-inverse* and is denoted by f^{-1}.

Fact.

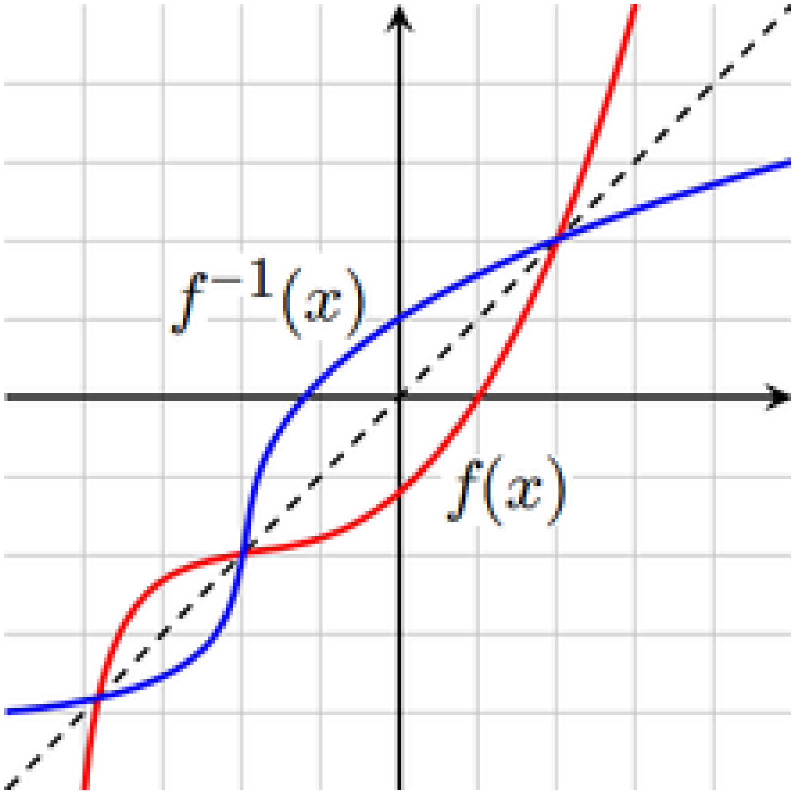
- f is the inverse for g iff g is the inverse for f.
- f is the inverse for g iff for every $x \in X$ and $y \in Y$

$$f(x)=y \Leftrightarrow g(y)=x$$

- f is invertible iff f is "one-to-one" and "onto".
- the graph of

Fact.

The graph of the inverse function of *f* is the mirror reflection of the graph of *f* in the line *y*=*x*. (Image from Wikipedia.)



ELEMENTARY FUNCTIONS

Definition.

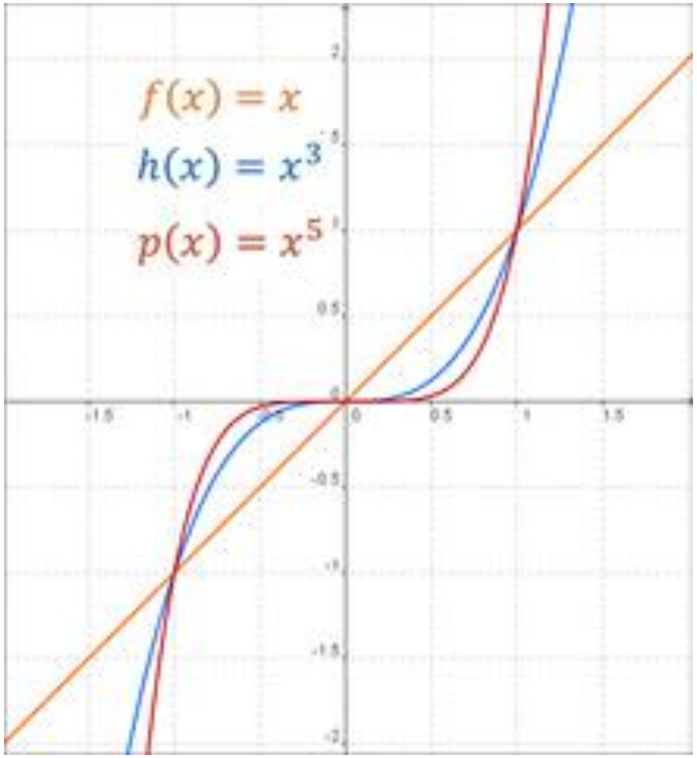
The set of *elementary* functions consists of:

- constant functions (constant on ℝ)
- *id* function (*id*(x) = x, the "do nothing" function
- trigonometric functions
- power functions (x^b where b is a real number, not necessarily an integer)
- exponential functions
- functions obtained by arithmetic operations on elementary functions (sums, product, quotients …)
- compositions of elementary functions
- inverses of elementary functions

Fact.

- Applying only addition and multiplication to constant functions and the identity function we get all polynomials.
- Applying division to polynomials we get rational functions.

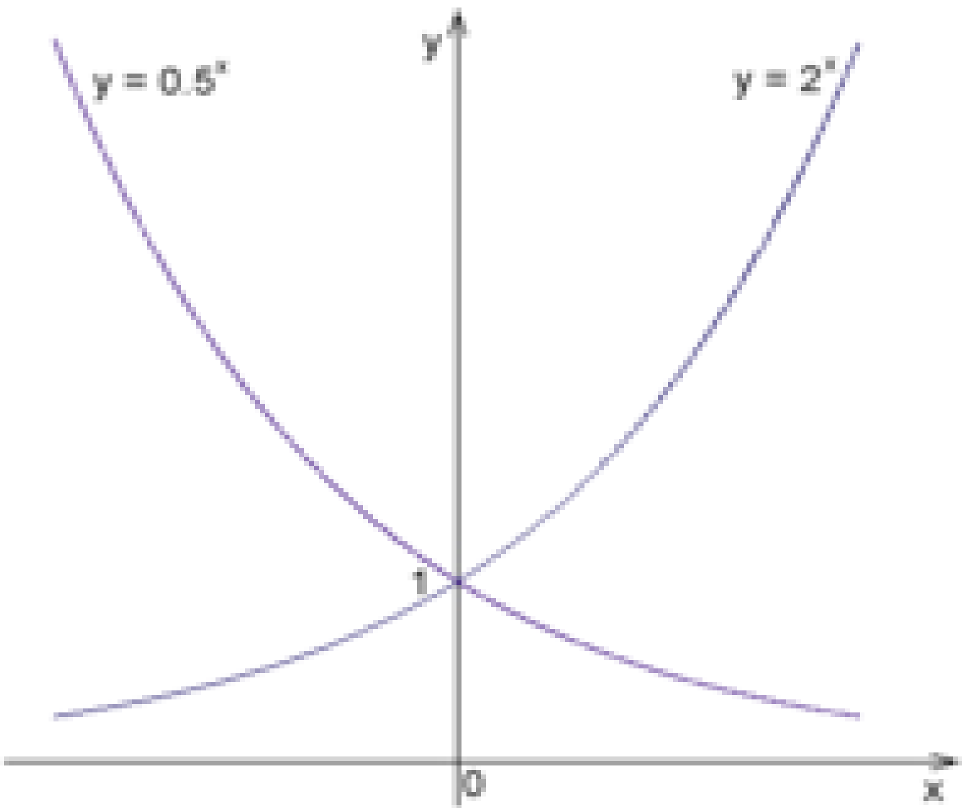
Power functions are functions of the form f(x) = x^a where *a* is a constant. If *a*=1, x^a becomes the identity function.



The image from Wikipedia

Exponential functions are functions of the form $f(x) = a^x$ where a is a positive constant different from 1.

If $a>1$ then a^x is increasing, otherwise it is decreasing.



The image from Wikipedia

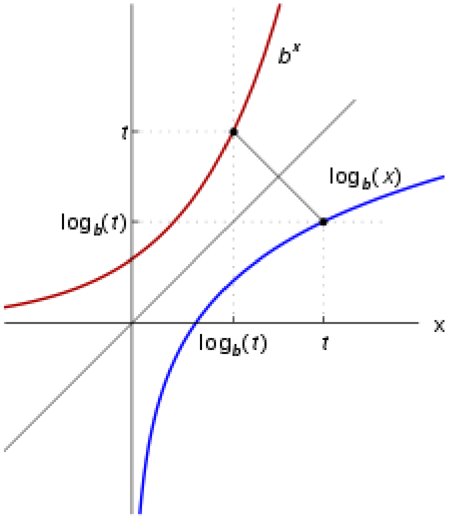
***Exponential functions* should not be confused with *power functions*.**

Exponential functions have a constant base, the variable, x, is in the exponent, as in 2^x.

In power functions the variable is in the base, the exponent is constant, as in x^2.

Fact. (Properties of powers)

- $a^b a^c = a^{b+c}$, this implies a^0=1 and $a^{-b} = \frac{1}{a^b}$
- $\left(a^b\right)^c = a^{bc}$
- $a^b c^b$=$(ac)^b$



The image from Wikipedia. In this example b>1.

Definition.

The inverse function to b^x is called the *logarithmic* function to base *b* and is denoted by $\log_b x$.

Fact.

- Since the domain of b^x is $\mathbb{R}$ and the set of values is $(0;\infty)$, the domain of log_b is $(0;\infty)$ and the range is $\mathbb{R}$.
- *b* must be positive and different from 1.

Examples.

1. $log_b b = 1$ for every b for which the expression makes sense
2. $log_b 1 = 0$
3. $log_{10} 100 = 2$
4. $log_{100} 10 = 0.5$
5. $log_2 1024 = 10$

Theorem. (Properties of the logarithmic function)

1. if $b>1$ then $log_b x$ is increasing, if $b<1$ then $log_b x$ is decreasing
2. $b^{\log_b x} = x$ and $\log_b b^x = x$
3. $log_b xy = log_b x + log_b y$, because $b^{\log_b xy} = xy = b^{\log_b x} b^{\log_b y}$ $= b^{log_b x + log_b y}$
4. The last formula implies $log_b \frac{x}{y} = log_b x - log_b y$
5. $log_b x^a = a\ log_b x$, because $b^{log_b x^a} = x^a = \left(b^{\log_b x}\right)^a = b^{a log_b x}$
6. (Change of base) $\frac{log_b x}{log_b y} = log_y x$

Theorem. (Properties of the logarithmic function).

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2. $b^{log_b x} = x$ and $\log_b b^x = x$
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Comprehension.

1. Prove property 2
2. Prove the change of base property
3. If (a_n) is a geometric sequence what type is $(log_b a_n)$?

Inverse trigonometric functions
source - Wikipedia

Name	Usual notation	Definition	Domain of x for real result	Range of usual principal value (radians)	Range of usual principal value (degrees)
arcsine	$y = \arcsin(x)$	$x = \sin(y)$	$-1 \leq x \leq 1$	$-\pi/2 \leq y \leq \pi/2$	$-90° \leq y \leq 90°$
arccosine	$y = \arccos(x)$	$x = \cos(y)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$	$0° \leq y \leq 180°$
arctangent	$y = \arctan(x)$	$x = \tan(y)$	all real numbers	$-\pi/2 < y < \pi/2$	$-90° < y < 90°$
arccotangent	$y = \operatorname{arccot}(x)$	$x = \cot(y)$	all real numbers	$0 < y < \pi$	$0° < y < 180°$

More about cyclometric functions in the old presentation.

Hyperbolic functions:

- Hyperbolic sine: sinh x=$\frac{e^x - e^{-x}}{2}$
- Hyperbolic cosine: cosh x =$\frac{e^x + e^{-x}}{2}$
- Hyperbolic tangent: tanh x = $\frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- Hyperbolic cotangent: for x $\neq$ 0, coth x = $\frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Fact.

Just like x=cos *t* and y=sin *t* satisfy the equation of the unit circle, $x^2+y^2=1$ so x=cosh *t* and y=sinh *t* satisfy the equation of the equilateral hyperbola, $x^2-y^2 = 1$.

In fact:

$$\left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{e^{2t}+2+e^{-2t}}{4} - \frac{e^{2t}-2+e^{-2t}}{4} = \frac{4}{4} = 1$$